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Approximation by Discrete Operators

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A discrete, positive, weighted algebraic polynomial operator which is based on Gaussian quadrature is constructed. The operator is shown to satisfy the Jackson estimate and an optimal version is obtained.

1. INTRODUCTION

There has been some recent interest [2, 8, 9] in obtaining discrete versions of positive linear integral operators. In this paper, we construct a sequence of discrete, positive, weighted algebraic polynomial operators, $\{K_n\}$, with the property

$$\| K_n(f) - f \| \leqslant C_1 \omega \left(f, \frac{1}{n}\right) + \frac{C_2 \|f\|}{n^2}, \tag{1.1}$$

for all n sufficiently large, where C_1 and C_2 are positive constants, independent of f. In (1.1) $f \in C[0, 1]$, the norms are taken over some subinterval $[a, b]$ of $(0, 1)$, and $\omega(f, \cdot)$ denotes the modulus of continuity of f on $[0, 1]$. The construction of K_n is based on an approach taken by Bojanic [1] and DeVore [3, Chap. 6].

Let $w(x)$ be an even, nonnegative, bounded, Lebesgue integrable function bounded on $[-1, 1]$ with the properties:

(i) on each interval $[a, b] \subseteq (-1, 1)$, there is $m > 0$ such that $0 < m \leqslant w(x)$, $x \in [a, b]$;

(ii) $w(x)$ has a continuous second derivative on $(-1, 1)$; and

(iii) $w(0) = 1$.

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Let $\{P_n(x)\}$ be the sequence of orthonormal polynomials on $[-1, 1]$ associated with $w(x)$. It is known [10, p. 44] that the zeros of $P_n(x)$ are real, simple, located in $(-1, 1)$ and symmetric about 0.

Let, for $0 \leqslant x \leqslant 1$, $\bar{w}(x) = 2w(2x - 1)$, and $Q_n(x) = P_n(2x - 1)$. Then $\{Q_n(x)\}$ is orthonormal on $[0, 1]$ with respect to $\bar{w}(x)$.

For each $n = 1, 2, \ldots$, let $x_{1n} < x_{2n} < \cdots < x_{nn}$ be the zeros of $Q_n(x)$. Let λ_{kn}, $k = 1, \ldots, n$, be the Cotes numbers corresponding to x_{kn}, $k = 1, \ldots, n$, so that, for each polynomial, p, of degree $\leqslant 2n - 1$,

$$\int_0^1 \bar{w}(x)\, p(x)\, dx = \sum_{k=1}^{n} \lambda_{kn}\, p(x_{kn})$$

by the Gauss quadrature rule [10, p. 47].

Let $y_{1n} < y_{2n} < \cdots < y_{nn}$ be the zeros of $P_n(x)$ and μ_{kn}, $k = 1, \ldots, n$, the corresponding Cotes numbers.

Define the control function $g = g(x, t)$ for $t \in [-1, 1]$ and $x \in (0, 1)$ by

$$g(x, t) = 1 + \left(\frac{\bar{w}'(x) - \bar{w}(x)\, w'(0)}{\bar{w}(x)}\right) t. \tag{1.2}$$

Assume $\bar{w}$ is such that, on some closed subinterval I, of $(0, 1)$, there is an $M > 0$ such that

$$0 < M \leqslant g(x, t), \qquad x \in I, \quad t \in [-1, 1]. \tag{1.3}$$

Let α_{2n} be the smallest positive zero of $P_{2n}(x)$ for each $n = 1, 2, \ldots$. Note that $-\alpha_{2n}$ is the largest negative zero of $P_{2n}(x)$. μ_{2n} denotes the Cotes number corresponding to α_{2n} and μ_{-2n} denotes the Cotes number corresponding to $-\alpha_{2n}$. Define, for $n = 1, 2, \ldots$,

$$R_n(t) = \left(\frac{P_{2n}(t)}{t^2 - \alpha_{2n}^2}\right)^2, \qquad t \in [-1, 1],$$

and

$$\frac{1}{C_n(x)} = \int_{-1}^{1} w(t)\, g(x, t)\, R_n(t)\, dt, \qquad x \in I. \tag{1.4}$$

For each $x \in I$, $g(x, t)\, R_n(t)$ is a polynomial in t of degree $4n - 3$. Hence, by the Gauss quadrature formula based on the zeros of $P_{2n}(t)$, we have

$$\frac{1}{C_n(x)} = \mu_{-2n}\, g(x, -\alpha_{2n})\, \gamma_{2n} + \mu_{2n}\, g(x, \alpha_{2n})\, \gamma_{2n}\,, \tag{1.5}$$

where γ_{2n} denotes the value of $(P_{2n}(t)/(t^2 - \alpha_{2n}2))^2$ at α_{2n}. Note that $C_n(x)$ is positive because of (1.3). For each $n = 1, 2,\ldots$, define the operator K_n by

$$K_n(f, x) = \frac{C_n(x)}{\overline{w}(x)} \sum_{K=1}^{2n} f(x_{K,2n})\, \lambda_{K,2n} R_n(x_{K,2n} - x), \tag{1.6}$$

where $f \in C[0, 1]$, $x \in I$, and $C_n(x)$ is given by (1.5). Clearly, K_n is a positive linear operator from $C[0, 1]$ to $C(I)$ and, except for the factor $C_n(x)/w(x)$, K_n is an algebraic polynomial.

It might seem more natural to consider the operator

$$K_n(f, x) = \frac{1}{w(x)} \sum_{K=1}^{2n} f(Y_{K,2n})\, \mu_{K,2n} R_n(Y_{K,2n} - x),$$

where x is restricted to a subinterval of $(-1, 1)$, $Y_{K,2n}$ are the zeros of $P_{2n}(x)$ on $[-1, 1]$ and

$$x_{K,2n} = \frac{1 + Y_{K,2n}}{2}.$$

This would avoid the shift from $[-1, 1]$ to $[0, 1]$. However, this shift is essential to the proof of Lemma 2.1 below. In particular, $x \in (0, 1)$ implies $-1 < -x < 1 - x < 1$ and hence (1.4) can be used in estimating $-1 + K_n(1, x)$ in the proof of Lemma 2.1.

2. Degree of Approximation

In the sequel, I denote the interval of (1.3).

Lemma 2.1. *There exists a positive constant, $C = C(w, I)$, such that for all n sufficiently large,*

$$\| K_n(1) - 1 \| \leqslant C/n^2,$$

where the norm is the sup norm over I and K_n is defined by (1.6).

Proof. By (1.6) and the Guass quadrature rule, we have, for $x \in I$,

$$\begin{aligned} K_n(1, x) &= \frac{C_n(x)}{\overline{w}(x)} \int_0^1 \overline{w}(t)\, R_n(t - x)\, dt \\ &= \frac{C_n(x)}{\overline{w}(x)} \int_{-x}^{1-x} \overline{w}(t + x)\, R_n(t)\, dt. \end{aligned}$$

Using (1.4) we obtain

$$\begin{aligned}
-1 + K_n(1, x) &= \frac{C_n(x)}{\bar{w}(x)} \int_{-x}^{1-x} R_n(t)(\bar{w}(t + x) - \bar{w}(x)\, w(t)\, g(x, t))\, dt \\
&\quad - C_n(x) \int_{-1}^{-x} w(t)\, g(x, t)\, R_n(t)\, dt \\
&\quad - C_n(x) \int_{1-x}^{1} w(t)\, g(x, t)\, R_n(t)\, dt \\
&= J_1 + J_2 + J_3 .
\end{aligned}$$

In view of (i), (ii), and (1.2) there is an absolute constant, M_1, such that

$$g(x, t) \leqslant M_1, \qquad t \in [-1, 1], \quad x \in I.$$

Next, since $x \in I$ is bounded away from both 0 and 1, there are positive constants M_2, M_3, which depend only on I such that

$$0 < M_2 \leqslant t^2, \qquad t \in [-1, -x], \quad x \in I,$$

and

$$0 < M_3 \leqslant t^2, \qquad t \in [1 - x, 1], \quad x \in I.$$

Hence

$$| J_2 | \leqslant \frac{C_n(x)}{M_2} \int_{-1}^{1} w(t)\, t^2 R_n(t)\, dt,$$

and

$$| J_3 | \leqslant \frac{C_n(x)}{M_3} \int_{-1}^{1} w(t)\, t^2 R_n(t)\, dt.$$

Using (1.3), (1.4), [3, Lemma 6.4], and [3, proof of Theorem 6.3], we obtain a constant $M_4 = M_4(I, w)$ such that,

$$C_n(x) \int_{-1}^{1} w(t)\, t^2 R_n(t)\, dt \leqslant M_4/n^2,$$

if n is sufficiently large. Consequently, for all n sufficiently large, there is a constant, M_5, such that

$$| J_2 | + | J_3 | \leqslant M_5/n^2, \qquad x \in I. \tag{2.1}$$

In view of (ii), (iii), and (1.2) the function

$$h(x, t) = \bar{w}(t + x) - \bar{w}(x)\, w(t)\, g(x, t)$$

satisfied, for each fixed $x \in I$,

$$h(x, 0) = 0,$$

and

$$\frac{\partial h}{\partial t}(x, 0) = 0.$$

By Taylor's formula, for each $x \in I$, there exist positive constants $M(x)$ and $\eta(x)$ such that

$$| h(x, t)| \leqslant M(x)\, t^2, \qquad | t | \leqslant \eta(x).$$

Since I is compact, we can find $\eta \in (0, 1)$ and $M_6 > 0$, both independent of $x \in I$, such that

$$| h(x, t)| \leqslant M_6 t^2, \qquad | t | \leqslant \eta, \quad x \in I.$$

We thus have, for $x \in I$,

$$\begin{aligned} \bar{w}(x) \mid J_1 \mid &\leqslant \int_{-\eta}^{\eta} C_n(x)\, R_n(t) \mid h(x, t)| \, dt \\ &\quad + \int_{-x}^{-\eta} C_n(x)\, R_n(t) \mid h(x, t)| \, dt \\ &\quad + \int_{\eta}^{1-x} C_n(x)\, R_n(t) \mid h(x, t)| \, dt \\ &= J_{11} + J_{12} + J_{13} \,. \end{aligned}$$

First,

$$\begin{aligned} J_{11} &\leqslant M_6 \int_{-\eta}^{\eta} C_n(x)\, t^2 R_n(t)\, dt \\ &\leqslant \frac{M_6}{m(\eta)} \int_{-\eta}^{\eta} C_n(x)\, w(t)\, t^2 R_n(t)\, dt \\ &\leqslant M_7 \int_{-1}^{1} C_n(x)\, w(t)\, t^2 R_n(t)\, dt, \end{aligned}$$

where $M_7 = M_6/m(\eta)$ and $m(\eta)$ is such that $0 < m(\eta) \leqslant w(t)$, $-\eta \leqslant t \leqslant \eta$. Hence, as in the proof of (2.1),

$$J_{11} \leqslant M_8/n^2, \qquad x \in I, \tag{2.2}$$

where M_8 is a constant.

Since $h(x, t)$ is globally bounded, say by M_9, we have

$$\begin{aligned} J_{12} &\leqslant M_9 \int_{-x}^{-\eta} C_n(x)\, R_n(t)\, dt \\ &\leqslant \frac{M_9}{\eta^2} \int_{-x}^{-\eta} C_n(x)\, t^2 R_n(t)\, dt \\ &\leqslant \frac{M_9}{M_{10}\eta^2} \int_{-x}^{-\eta} C_n(x)\, w(t)\, t^2 R_n(t)\, dt, \end{aligned}$$

where M_{10} is a constant guaranteed by (i). The argument used in establishing (2.1) yields a constant, M_{11}, such that

$$J_{12} \leqslant \frac{M_{11}}{\eta^2 n^2}, \qquad x \in I. \tag{2.3}$$

In a similar fashion, there is a constant, M_{12}, such that, for $x \in I$,

$$J_{13} \leqslant \frac{M_{12}}{\eta^2 n^2}. \tag{2.4}$$

Since $\overline{w}(x)$ is bounded away from 0 for $x \in I$, there exists a constant, M_{13}, such that, for $x \in I$,

$$| J_1 | \leqslant M_{13}/n^2. \tag{2.5}$$

Combining (2.5) and (2.1) completes the proof of Lemma 2.1.

LEMMA 2.2. *There exists a positive constant, $D = D(w, I)$, such that for all n sufficiently large,*

$$\| K_n((t - x^2); x)\| \leqslant \frac{D}{n^2},$$

where the norm is the sup norm taken over I.

Proof. Using (1.6) and the Gauss quadrature rule we obtain, for $x \in I$,

$$\begin{aligned} K_n((t - x)^2, x) &= \frac{C_n(x)}{\overline{w}(x)} \int_0^1 \overline{w}(t)(t - x)^2\, R_n(t - x)\, dt \\ &= \frac{C_n(x)}{\overline{w}(x)} \int_{-x}^{1-x} \overline{w}(t + x)\, t^2 R_n(t)\, dt \\ &\leqslant \frac{D_1 C_n(x)}{\overline{w}(x)\, D_2} \int_{-x}^{1-x} w(t)\, t^2 R_n(t)\, dt, \end{aligned}$$

where D_1, D_2 are constants guaranteed by the definition of w. Furthermore

$\bar{w}(x)$ is bounded away from 0 for $x \in I$. Thus, by the argument used in establishing (2.1) there is a constant, D, such that, for n sufficiently large,

$$K_n((t-x)^2, x) \leqslant D\left(\frac{1}{n^2}\right).$$

This completes the proof of Lemma 2.2.

We can now establish the main result of this paper.

THEOREM 2.3. *Let K_n be given by* (1.6) *and $f \in C[0, 1]$. Then, for all n sufficiently large,*

$$\| K_n(f) - f \| \leqslant C_1 \omega\left(f, \frac{1}{n}\right) + C_2 \cdot \| f \| \cdot \frac{1}{n^2},$$

where C_1 and C_2 are positive constants which depend only on the choice of w, the sup norm is taken over I, and $\omega(f, \cdot)$ denotes the modulus of continuity of f on [0, 1].

Proof. Using an inequality of Shisha and Mond [5], we have

$$\| K_n(f) - f \| \leqslant (\| K_n(1) \| + 1)\, \omega(f, \beta_n) + \| f \| \cdot \| K_n(1) - 1 \|,$$

where

$$\beta_n{}^2 = \| K_n((t-x)^2; x) \|.$$

The proof of the theorem is now an immediate consequence of Lemma 2.1 and Lemma 2.2.

The following two special cases of the operators defined by (1.6) are of interest.

EXAMPLE 2.4. If $w(x) \equiv 1$, then the orthonormal sequence $\{P_n(x)\}$ is the sequence of Legendre polynomials. In this case $g(x, t) \equiv 1$ and I can be any closed subinterval of (0, 1).

Using [10, p. 48] it can be shown that $\mu_{kn} = \lambda_{kn}$ and μ_{kn} is given by

$$\mu_{kn} = 2(1 - y_{kn}^2)^{-1}(p_n'(y_{kn}))^{-2}$$

([10, p. 352]).

Thus, in this case K_n is defined by

$$K_n(f, x) = \frac{C_n(x)}{2} \sum_{k=1}^{2n} f(x_{k,2n})\, \lambda_{k,2n} R_n(x_{k,2n} - x),$$

where

$$x_{k,2n} = \frac{1 + y_{k,2n}}{2}.$$

Estimates for $y_{k,2n}$ can be found in [10, p. 122].

EXAMPLE 2.5. If $w(x) = (1 - x^2)^{1/2}$, then the orthonormal sequence is the sequence, $\{u_n(x)\}$, of Chebyshev polynomials of the second kind. In this case $\bar{w}(x) = 2(x(1 - x))^{1/2}$. Elementary computations show that (1.3) is satisfied if I is any closed subinterval of $\{x: | x - \frac{1}{2} | < (2^{1/2} - 1)/2\}$.

For this case, the Cotes numbers, λ_{kn}, are given by ([10, p. 353])

$$\lambda_{kn} = \frac{\pi}{n+1} \sin^2 \left(\frac{n-k-1}{n+1} \pi \right),$$

and

$$x_{kn} = \frac{\left[1 + \cos \left(\frac{n-k-1}{n+1} \pi \right)\right]}{2}$$

$$= \cos^2 \left(\frac{n-k-1}{2(n+1)} \pi \right), \qquad k = 1, 2,..., n.$$

In this example the operator K_n defined by (1.6) takes a particularly convenient form.

The operator (1.6) is essentially a discrete version of the convolution operator

$$L_n(f, x) = \int_0 f(t) R_n(t - x) \, dt, \qquad 0 \leqslant x \leqslant 1, \tag{2.6}$$

where

$$R_n(t) = C_n \left(\frac{P_{2n}(t)}{t^2 - \alpha_{2n}^2} \right)^2, \qquad -1 \leqslant t \leqslant 1,$$

and

$$\int_{-1}^{1} R_n(t) \, dt = 1, \qquad n = 1, 2, 3,... .$$

Approximation in the space $L_p[0, 1]$ via (2.6) is to be considered in [6]; (2.6) is close to the method Λ_n of Bojanic [1] and DeVore [3, Chap. 6]. DeVore has shown that Λ_n is optimal in a certain sense.

We can attain optimality for a version of our discrete operators, using a constant weight (Legendre polynomials). Specifically, let $w(x) \equiv 1$ be our weight function on $[-1, 1]$ and let $\{P_n(x)\}$ be the associated orthonormal Legendre polynomials. For $0 \leqslant x \leqslant 1$, let $\bar{w}(x) = 2w(2x - 1) \equiv 2$ and

$Q_n(x) = P_n(2x - 1)$ be the shifted Legendre polynomials. Hence $\{Q_n(x)\}$ is orthonormal on $[0, 1]$ with weight $\bar{w}(x) = 2$. For each $n = 1, 2,\ldots,$ let $x_{1n} < x_{2n} < \cdots < x_{nn}$ be the zeros of $Q_n(x)$ and let λ_{kn}, $k = 1, 2,\ldots, n$, be the associated Cotes numbers (recall Example (2.4). Let $0 < \alpha_{2n} < \alpha_{2n-1}$ be the two smallest positive zeros of $P_{2n}(x)$ for each $n = 2, 3,\ldots$. Notice that $0 > -\alpha_{2n} > -\alpha_{2n-1}$ are the two largest negative zeros of $P_{2n}(x)$. Define, for $n = 2, 3,\ldots$ and $-1 \leqslant t \leqslant 1$,

$$R_n(t) = C_n \left[\frac{P_{2n}(t)}{(t^2 - \alpha_{2n}^2)(t^2 - \alpha_{2n-1}^2)}\right]^2,$$

where $C_n > 0$ is chosen so that

$$\int_{-1}^{1} R_n(t)\, dt = 1, \qquad n = 2, 3,\ldots .$$

Hence (recall example 2.4),

$$\begin{aligned}\frac{1}{C_n} &= \int_{-1}^{1} \left[\frac{P_{2n}(t)}{(t^2 - \alpha_{2n}^2)(t^2 - \alpha_{2n-1}^2)}\right]^2 dt \\ &= \sum_{k=1}^{2n} \lambda_{k,2n} \left[\frac{P_{2n}(Y_{k,2n})}{(Y_{k,2n}^2 - \alpha_{2n}^2)(Y_{k,2n}^2 - \alpha_{2n-1}^2)}\right]^2,\end{aligned}$$

where $Y_{k,2n} = 2x_{k,2n} - 1$, $k = 1, 2,\ldots, 2n$.

Therefore, for $n = 2, 3,\ldots,$

$$1/C_n = (\lambda_{-2n,2n} + \lambda_{2n,2n})\, \gamma_{2n} + (\lambda_{-(2n-1),2n-1} + \lambda_{2n-1,2n-1})\, \gamma_{2n-1},$$

where γ_{2n-i} denotes the value of

$$\left\{\frac{P_{2n}(t)}{(t^2 - \alpha_{2n}^2)(t^2 - \alpha_{2n-1}^2)}\right\}^2$$

at $t = \alpha_{2n-i}$, $i = 0, 1$.

Let $I_\delta = \{x: 0 < \delta \leqslant x \leqslant 1 - \delta < 1\}$. For $f \in C[0, 1]$, $x \in I_\delta$ and $n = 2, 3,\ldots,$ define

$$K_n(f, x) = \frac{1}{2} \sum_{k=1}^{2n} \lambda_{k,2n} f(x_{k,2n})\, R_n(x_{k,2n} - x). \tag{2.7}$$

Thus $K_n(f, x)$ is a positive linear operator from $C[0, 1]$ to $C(I_\delta)$ and $K_n(f, x)$ is an algebraic polynomial of degree $\leqslant 4n - 8$.

LEMMA 2.6. *For $x \in I_\delta$ and all n sufficiently large,*

$$| K_n(1, x) - 1 | \leqslant C_\delta \frac{1}{n^4},$$

where $C_\delta > 0$ is a constant.

Proof. We follow the lines of the proof of Lemma 2.1. By Gauss quadrature, for $x \in I_\delta$,

$$\begin{aligned} K_n(1, x) &= \frac{1}{2} \sum_{k=1}^{2n} \lambda_{k,2n} R_n(x_{k,2n} - x) \\ &= \frac{1}{2} \int_0^1 2R_n(t - x)\, dt \\ &= \int_{-x}^{1-x} R_n(t)\, dt. \end{aligned}$$

Hence

$$\begin{aligned} K_n(1, x) - 1 &= \int_{-x}^{1-x} (R_n(t) - R_n(t))\, dt \\ &\quad - \int_{-1}^{-x} R_n(t)\, dt - \int_{1-x}^{1} R_n(t)\, dt \\ &= - \int_{-1}^{-x} R_n(t)\, dt - \int_{1-x}^{1} R_n(t)\, dt \\ &= J_2 + J_3. \end{aligned}$$

Since $x \in I_\delta$,

$$\begin{aligned} | J_2 | &\leqslant \frac{1}{\delta^4} \int_{-1}^{-x} t^4 R_n(t)\, dt \\ &\leqslant \frac{1}{\delta^4} \int_{-1}^{-1} t^4 R_n(t)\, dt. \end{aligned}$$

Using the fact that degree of $R_n(t)$ is $4n - 8$ and [3, proof of Theorem 6.2], we find a constant $M_1 > 0$ such that

$$| J_2 | \leqslant \frac{M_1}{\delta^4} \cdot \frac{1}{n^4}.$$

A similar estimate holds for $| J_3 |$. Hence there is a constant $C_\delta > 0$ such that

$$| K_n(1, x) - 1 | \leqslant \frac{C_\delta}{n^4}, \qquad x \in I_\delta.$$

Let $e_i(t) = t^i$, $i = 0, 1, 2$. The optimal operators of DeVore [3, p. 171] are defined as follows.

A sequence of positive, algebraic, polynomial operators L_n is said to be optimal on $[c, d]$, if L_n maps $C[a, b]$ into $C[c, d]$ and for $i = 0, 1$

$$\| e_i - L_n(e_i)\| = o(n^{-2})$$

while for e_2,

$$\| e_2 - L_n(e_2)\| = O(n^{-2}).$$

THEOREM 2.7. *Operators* (2.7) *are optimal on* $I_\delta = \{x : 0 < \delta \leqslant x \leqslant 1 - \delta < 1\}$.

Proof. We have

$$\| K_n(e_0) - e_0 \| = o(n^{-2}) \tag{2.8}$$

by Lemma 2.6. Let $x \in I_\delta$. Then

$$\begin{aligned}
| K_n(e_1, x) - x | &= \left| x - \frac{1}{2} \sum_{k=1}^{2n} \lambda_{k,2n} R_n(x_{k,2n} - x)\, x_{k,2n} \right| \\
&= \left| x \int_{-1}^{1} R_n(t)\, dt - \int_{-x}^{1-x} (t + x)\, R_n(t)\, dt \right| \\
&= \left| x \int_{-x}^{1-x} R_n(t)\, dt + x \int_{-1}^{-x} R_n(t)\, dt + x \int_{1-x}^{1} R_n(t)\, dt \right. \\
&\quad \left. - x \int_{-x}^{1-x} R_n(t)\, dt - \int_{-x}^{1-x} tR_n(t)\, dt \right| \\
&\leqslant \left| x \int_{-1}^{-x} R_n(t)\, dt \right| + \left| x \int_{1-x}^{1} R_n(t)\, dt \right| + \left| \int_{-x}^{1-x} tR_n(t)\, dt \right| \\
&= | J_1 | + | J_2 | + | J_3 |.
\end{aligned}$$

By Lemma 2.6, there is a positive constant $M_1(\delta)$, such that

$$| J_i | \leqslant \frac{M_1(\delta)}{n^4}, \qquad i = 1, 2.$$

Next, since $R_n(t)$ is even,

$$\begin{aligned}
| J_3 | = \left| \int_{-x}^{1-x} tR_n(t)\, dt \right| &= \left| \int_{-x}^{0} tR_n(t)\, dt + \int_{0}^{1-x} tR_n(t)\, dt \right| \\
&= \left| - \int_{0}^{x} tR_n(t)\, dt + \int_{0}^{1-x} tR_n(t)\, dt \right| \\
&= \left| \int_{x}^{1-x} tR_n(t)\, dt \right|
\end{aligned}$$

and, as in Lemma 2.6, this last integral can be estimated by

$$\left| \frac{1}{\delta^3} \int_x^{1-x} t^4 R_n(t)\, dt \right| \leqslant \frac{1}{\delta^3} \int_{-1}^{1} t^4 R_n(t)\, dt \leqslant \frac{M_2}{\delta^3} \cdot \frac{1}{n^4},$$

where $M_2 > 0$ is a constant. Hence

$$| K_n(e_1, x) - x | \leqslant \frac{M_3(\delta)}{n^4} \tag{2.9}$$

for a positive constant $M_3(\delta)$ if $x \in I_\delta$.

Finally, for $x \in I_\delta$,

$$\begin{aligned} | K_n(e_2, x) - x^2 | &< | K_n(e_2, x) - x^2 K_n(e_0, x)| \\ &\quad + | x^2(K_n(e_0, x) - 1)| \\ &\leqslant | K_n(e_2 - x^2 e_0, x)| + \frac{M_4(\delta)}{n^4}, \end{aligned}$$

by Lemma 2.6, for some constant $M_4(\delta)$.

Now

$$\begin{aligned} K_n(e_2 - x^2 e_0, x) &= \frac{1}{2} \sum_{k=1}^{2n} \lambda_{k,2n}(x_{k,2n}^2 - x^2)\, R_n(x_{k,2n} - x) \\ &= \int_0^1 (t^2 - x^2)\, R_n(t - x)\, dt \\ &= \int_{-x}^{1-x} [(t + x)^2 - x^2]\, R_n(t)\, dt \\ &= \int_{-x}^{1-x} (t^2 + 2tx)\, R_n(t)\, dt. \end{aligned}$$

Thus

$$| K_n(e_2 - x^2 e_0, x)| \leqslant \left| \int_{-x}^{1-x} t^2 R_n(t)\, dt \right| + \left| 2x \int_{-x}^{1-x} t R_n(t)\, dt \right|.$$

As above, for a constant $M_5(\delta)$,

$$\left| 2x \int_{-x}^{1-x} t R_n(t)\, dt \right| \leqslant \frac{M_5(\delta)}{n^4}$$

and

$$\left| \int_{-x}^{1-x} t^2 R_n(t)\, dt \right| \leqslant \int_{-1}^{1} t^2 R_n(t)\, dt \leqslant \frac{M_6}{n^2},$$

for some positive constant M_6, as in [3, proof of Theorem 6.2]. Thus, for $x \in I_\delta$,

$$|K_n(e_2, x) - x^2| \leqslant \frac{M_7(\delta)}{n^2} \tag{2.10}$$

for a positive constant $M_7(\delta)$. Optimality of (2.7) follows from (2.8), (2.9), and (2.10).

Saturation and related topics for (2.7) have been discussed in [7].

It is possible to modify (1.6) so as to obtain the estimate of Theorem 2.3, minus the term $\|f\|/n^2$, on all of [0, 1].

Let g denote the linear map which takes [0, 1] onto I. Then g^{-1} maps I onto [0, 1] and [0, 1] onto a larger interval, say $[c, d]$. For $f \in C[0, 1]$ define $f(x) = f(0)$, $c \leqslant x \leqslant 0$ and $f(x) = f(1)$, $1 \leqslant x \leqslant d$. Then $f \circ g^{-1} \in C[0, 1]$ and we define the projection P_1 from $C[0, 1]$ to the constants by $P_1(h) = h(0)$ for $h \in C[0, 1]$. Let $\alpha = P_1(f \circ g^{-1}) = f \circ g^{-1}(0)$ and define the linear operator

$$L_n(f, x) = K_n(f \circ g^{-1} - \alpha, g(x)) + \alpha, \qquad 0 \leqslant x \leqslant 1. \tag{2.11}$$

THEOREM 2.8. *For $f \in C[0, 1]$ and defined as above on $[c, d] = g^{-1}[0, 1]$, $0 \leqslant x \leqslant 1$, and all n sufficiently large,*

$$|L_n(f, x) - f(x)| \leqslant T_1 \omega\left(f, \frac{1}{n}\right),$$

where T_1 is a positive constant which depends only on the choice of the weight function w and $\omega(f, \cdot)$ denotes the modulus of continuity of f on [0, 1].

Proof. Let $x \in [0, 1]$. Since $g(x) \in I$ and $f \circ g^{-1} - \alpha \in C[0, 1]$, using Theorem 2.3, (1.6), and the definition of f on $[c, d]$, we obtain

$$\begin{aligned} |L_n(f, x) - f(x)| &= |K_n(f \circ g^{-1} - \alpha, g(x)) - (f \circ g^{-1}(g(x)) - \alpha)| \\ &\leqslant C_1 \omega\left(f \circ g^{-1} - \alpha, \frac{1}{n}\right) + \frac{C_2 \|f \circ g^{-1} - \alpha\|}{n^2} [0, 1] \\ &= C_1 \omega\left(f, \frac{1}{n}\right) + \frac{C_2 \|f \circ g^{-1} - \alpha\|}{n^2} [0, 1], \end{aligned} \tag{2.12}$$

where $\omega(f, \cdot)$ is the modulus of continuity of f on [0, 1].

By [4, Corollary 3.1] and [11],

$$\|f \circ g^{-1} - \alpha\|_{[0,1]} \leqslant C_3 \omega(f \circ g^{-1}, 1), \tag{2.13}$$

where $C_3 > 0$ is a constant.

Theorem 2.8 follows from (2.12), (2.13), the definition of f, and properties of the modulus of continuity.

Notice that (2.11) is discrete but is not a positive operator.

REFERENCES

1. R. BOJANIC, A note on the degree of approximation to continuous functions, *Enseignement Math.*, *T* **15** (1969), 43–51.
2. R. BOJANIC AND O. SHISHA, Approximation of continuous, periodic functions by discrete positive linear operators, *J. Approximation Theory* **11** (1974), 231–235.
3. R. A. DEVORE, "The Approximation of Continuous Functions by Positive Linear Operators," Lecture Notes in Mathematics No. 293, Springer-Verlag Berlin/Heidelberg/New York, 1972.
4. R. A. DEVORE, Degree of approximation, *in* "Approximation Theory II" (G. G. Lorentz, C. K. Chui, and L. L. Schumaker, Eds.), pp. 117–159, Academic Press, New York, 1976.
5. O. SHISHA AND B. MOND, The degree of convergence of sequences of linear positive operators, *Proc. Nat. Acad. Sci. USA* **60** (1968), 1196–1200.
6. J. SWETITS AND B. WOOD, Degree of L_p-approximation by positive convolution operators, in preparation.
7. J. SWETITS AND B. WOOD, A note on the degree of approximation with an optimal, discrete polynomial, preprint.
8. J. SZABADOS, "Convergence and Saturation Problems of Discrete Linear Operators," I.S.N.M., Vol. 25, Birkhäuser, Basel, 1976, 405–419.
9. J. SZABADOS, On an interpolatory analog of the de la Vallee–Poussin means, *Studia Sci. Math. Hungar.* **9** (1974), 189–191.
10. G. SZEGO, "Orthogonal Polynomials," American Mathematical Society Colloquim Publication, Vol. XXIII, Providence, R.I., 1967.
11. H. WHITNEY, On functions with n-th bounded differences, *J. Math. Pures Appl.* **36** (1957), 67–95.